Maths Department	**Math 241**	**6 pages**
14/7/1438H	**Exam 1**	**90 minutes [25 marks]**

Name:

Student ID:

Student Section:

Serial number:

Marks: $\dfrac{\quad}{25}$

Q1: Mark True (T) or False (F) and justify your answers(5 marks):

(1) [F] If one row in an echelon form of an augmented matrix is $\begin{bmatrix} 0 & 0 & 0 & 5 & 0 \end{bmatrix}$ then the associated linear system is inconsistent.

$\begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

(2) [T] If A is $n \times n$ matrix, then $A - A^T$ is skew-symmetric.

$(A - A^T)^T = A^T - A = -(A - A^T)$

(3) [F] If $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 4 & 2 \\ 9 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix}$

(4) [T] If AA^T is singular matrix, then A is also singular.

$|AA^T| = 0 \Rightarrow |A||A^T| = 0 \Rightarrow |A|^2 = 0 \Rightarrow |A| = 0$

A is singular

(5) [T] If A is an invertible matrix, then for any matrix B; $|A^{-1}BA| = |B|$.

$|A^{-1}BA| = |A^{-1}||B||A| = \frac{1}{\cancel{|A|}} |B| \cancel{|A|} = |B|$

Q2: Fill in the blanks (5 marks):

(1) If A is 3×3 matrix such that $|A| = 9$, then $|3A^{-1}| = $ .3.

$$|3A^{-1}| = 3^3 |A^{-1}| = 3^3 \frac{1}{|A|} = \frac{3^3}{9} = 3$$

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(2) If $A \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$

$$A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

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(3) $\begin{vmatrix} \dots & 2 & 1 \\ 0 & \dots & -1 \\ 0 & 0 & 5 \end{vmatrix} = 15$

$(-1)\ (-3)$ or $(1)\ (3)$

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(4) If $A = [a_{ij}]$ is $n \times n$ skew-symmetric matrix, then $a_{ii} = 0$ $\forall i = 1, 2, \dots, n$

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(5) The system $\begin{aligned} x + y - 2z &= 1 \\ 3x + 3y - 6z &= 2 \end{aligned}$ has .no solution

(6) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 3 & -2 & -20 \end{bmatrix}$, then $Trace(A) = .-20$

(7) $A = \begin{bmatrix} k-1 & 2 \\ 4 & k+1 \end{bmatrix}$ is singular if $k = \ldots$

$|A| = k^2 - 1 - 8 = 0$

$k^2 = 9 \Rightarrow k = \pm 3$

Q3: For what values of k the following system has:

(a) No solution.

(b) An infinite number of solutions.

(c) Exactly one solution.

$[2\tfrac{1}{2}]$

$$x + 2y - z = 3$$
$$-x - y + z = 2$$
$$-x + y + z = k$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ -1 & -1 & 1 & 2 \\ -1 & 1 & 1 & k \end{array}\right] \xrightarrow[R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 3 & 0 & k+3 \end{array}\right]$$

$$\xrightarrow{-3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & k-12 \end{array}\right]$$

(a) $k - 12 \neq 0 \Rightarrow k \neq 12$

(b) $k - 12 = 0 \Rightarrow k = 12$

(c) Not possible.

Q4: Let A be an investable $n \times n$ matrix, prove that:

(a) $AB = AC \Rightarrow B = C$

$AB = AC$

$A^{-1}AB = A^{-1}AC$ since (A is invertabe)

$\Rightarrow IB = IC \Rightarrow B = C$

(b) $|A^{-1}| = \dfrac{1}{|A|}$

Since A is invertable, $AA^{-1} = I \Rightarrow |AA^{-1}| = |I|$

$\Rightarrow |A|\,|A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$ when $|A| \neq 0$ (A is invertable)

(c) If A is orthogonal, then $|A| = \pm 1$

$A^T = A^{-1}$ (A is orthogonal)

$AA^{-1} = I$ (A is invertable)

$AA^T = I \Rightarrow |AA^T| = 1 \Rightarrow |A||A^T| = |A|^2 = 1 \Rightarrow |A| = \pm 1$

Q5: Verify that the equation $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - \begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$$

$$= bc^2 - b^2c - (ac^2 - a^2c) + ab^2 - a^2b$$

$$=$$

Q6: Use Cramer's rule to solve the system,
$$5x + 4y = 2$$
$$-x + y = -22$$

$x = \dfrac{|A_1|}{|A|}$, $y = \dfrac{|A_2|}{|A|}$

$|A| = \begin{vmatrix} 5 & 4 \\ -1 & 1 \end{vmatrix} = 5 + 4 = 9$

$|A_1| = \begin{vmatrix} 2 & 4 \\ -22 & 1 \end{vmatrix} = 2 + 4(22) = 90$; $|A_2| = \begin{vmatrix} 5 & 2 \\ -1 & -22 \end{vmatrix} = -115 + 2$
$= -113$

$\Rightarrow x = 90/9 = 10$, $y = -113/9$

Q7: Find A^{-1} by using the adjoint matrix, where $A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{bmatrix}$

The cofactor is given by:

~~Ad~~

$$\begin{bmatrix} \begin{vmatrix} 2 & 5 \\ -5 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 8 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 8 & -5 \end{vmatrix} \\ -\begin{vmatrix} -2 & 3 \\ -5 & -2 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 8 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 8 & -5 \end{vmatrix} \\ \begin{vmatrix} -2 & 3 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 2 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 44 & -26 \\ -19 & -36 & 26 \\ -16 & -14 & 12 \end{bmatrix}$$

$$Adj = \begin{bmatrix} 21 & -19 & -16 \\ 44 & -36 & -14 \\ -26 & 26 & 12 \end{bmatrix}$$

$|A| = 4 \begin{vmatrix} 2 & 5 \\ -5 & -2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 5 \\ 8 & -2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 2 \\ 8 & -5 \end{vmatrix} = 4(21) + 2(-44) + 3(-26)$
$= -82$

$$A^{-1} = \frac{1}{-82} \begin{bmatrix} 21 & -19 & -16 \\ 44 & -36 & -14 \\ -26 & 26 & 12 \end{bmatrix}$$